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Existence Results for Primal and Dual Generalized Vector Equilibrium Problems With Applications to Generalized Semi-Infinite Programming

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Abstract. In this paper, we introduce several kinds of maximal pseudomonotonicity and establish existence theorems of maximal pseudomonotonicity. From these results we establish the existence theorems of generalized vector equilibrium problems. We establish existence theorems of generalized vector semi-infinite programming, as applications of generalized vector equilibrium problems.

Key words: Generalized vector equilibrium problem, Generalized vector semi-infinite programming, Maximal pseudomonotone, Pseudomonotone, C_x-quasiconvex

1. Introduction

Let X be a nonempty subset of a topological vector space E, $f{:}X\times X\to\mathbb{R}$ be a function such that $f(x,x)\geqslant 0$ for all $x\in X$, the equilibrium problem (in short EP) is to find $\bar{x}\in X$ such that

$$f(\bar{x},y)\geqslant 0 \quad \text{for all } y\in X. \tag{1}$$

This problem contains optimization, Nash equilibrium, fixed point, complementarity, variational inequality problems, and many others as special cases; for detail, see [6].

If Z is a topological vector space and $C:X\to 2^Z$ such that for each $x\in X$, $C(x)$ is a closed convex cone with int $C(x)\neq\emptyset$, then (1) can be generalized in the following ways:
Find $\bar{x}\in X$ such that

$$f(\bar{x},y)\notin -\mathrm{int}C(\bar{x}) \quad \text{for all } y\in X. \tag{2}$$

Find $\bar{x}\in X$ such that

$$f(\bar{x},y)\in C(\bar{x}) \quad \text{for all } y\in X. \tag{3}$$

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In this case, (EP) is called vector equilibrium problem (in short VEP) which also contains many problems as special cases. For further detail on (VEP) of form (2), we refer to [10] and references therein.

If $F:X \times X -\circ Z$ is a multivalued map, we consider the following four types of generalized vector equilibrium problems:
Find $\bar{x} \in X$ such that

$$F(\bar{x}, y) \nsubseteq -\mathrm{int}C(\bar{x}) \quad \text{for all } y \in X; \tag{4}$$

find $\bar{x} \in X$ such that

$$F(\bar{x}, y) \subseteq C(\bar{x}) \quad \text{for all } y \in X; \tag{5}$$

find $\bar{x} \in X$ such that

$$F(\bar{x}, y) \cap C(\bar{x}) \neq \emptyset \quad \text{for all } y \in X; \tag{6}$$

and find $\bar{x} \in X$ such that

$$F(\bar{x}, y) \cap (-\mathrm{int}C(\bar{x})) = \emptyset \quad \text{for all } y \in X. \tag{7}$$

If F is a single-valued function, then problems (4) and (7) are reduced to problem (2), and problem (5) and (6) are reduced to problem (3). Problems (4)–(7) contain generalized implicit vector variational inequalities, generalized vector variational inequality, and variational like inequality problems as special cases; See for example [10]. Problem (4) is considered and studied in [2,3,14,19]. Problems (4)–(7) are considered and studied in [1], and problem (5) is considered and studied in [8] when $C(x) = C$ for all $x \in X$. If $G:X \times X -\circ Z$ and $D: X -\circ Z$ are multivalued maps, a problem which is closely related to (GVEP) (4) is to find $\bar{x} \in X$ such that

$$G(y, \bar{x}) \nsubseteq \mathrm{int}D(\bar{x}) \quad \text{for all } y \in X. \tag{8}$$

A problem which is closely related to (GVEP) (5) is to find $\bar{x} \in X$ such that

$$G(y, \bar{x}) \subseteq -D(\bar{x}) \quad \text{for all } y \in X. \tag{9}$$

A problem which is closely related to (GVEP) (6) is to find $\bar{x} \in X$ such that

$$G(y, \bar{x}) \cap (-D(\bar{x})) \neq \emptyset \quad \text{for all } y \in X. \tag{10}$$

A problem which is closed related to (GVEP) (7) is to find $\bar{x} \in X$ such that

$$G(y, \bar{x}) \cap (\mathrm{int}D(\bar{x})) = \emptyset \quad \text{for all } y \in X. \tag{11}$$

Throughout this paper, we denote WIK^p, SIK^p, SIIK^p, WIIK^p, $\mathrm{WIK}^d_{G,D}$, $\mathrm{SIK}^d_{G,D}$, $\mathrm{SIIK}^d_{G,D}$, and $\mathrm{WIIK}^d_{G,D}$, the solutions sets of (GVEP) (4), (GVEP) (5), (GVEP) (6), (GVEP) (7), (DGVEP) (8), (DGVEP) (9), (DGVEP) (10), and (DGVEP) (11) respectively.

The following problems are special cases of problems (4)–(7) and (8)–(11).

(i) Let X be nonempty subset of $\mathbb{R}^n$ and let $\langle .,. \rangle : X \times X \to \mathbb{R}$ be the inner product in $\mathbb{R}^n$. Let $T:X \to \mathbb{R}^n$ be a function. Let $F: X \times X \to \mathbb{R}$ and $G:X \times X \to \mathbb{R}$ be defined by

$$F(x, y) = \langle T(x), y - x \rangle, G(y, x) = \langle T(y), x - y \rangle.$$

Then problems (4)–(7) are the same problem that is the Stampachia variational inequality problem [17]. And Problems (8)–(11) are the same problem that is the Minty variational inequality problem [17].

(ii) If X is a nonempty subset of a Banach space E and if $T:X -\!\circ E^*$ is a given multivalued map, where E^* denotes the dual space of E, let $F:X \times X \to \mathbb{R}$ and $G:X \times X \to \mathbb{R}$ be defined by

$$F(x, y) = \sup_{u \in T(x)} \langle u, y - x \rangle, G(y, x) = \inf_{v \in T(y)} \langle u, y - x \rangle.$$

Then problems (4)–(7) are the same problem that is the Stampachia variational inequality problem. And (8)–(11) are the same problem that is the Minty variational inequality problem. In [19], we study these two types of problems.

(iii) Let Z be a t.v.s., $D, C:X -\!\circ Z$ are multivalued maps such that for each $x \in X, C(x)$ and $D(x)$ are closed pointed convex cones with nonempty interior $\mathrm{int}C(x)$. Let X be a nonempty subset of a t.v.s. E, $H: X \to Z$, $T: X -\!\circ L(E, Z)$, where $L(E, Z)$ denotes the space of all continuous linear operator form E to Z. For x, $y \in X$, $t \in T(x)$, $\langle t, y \rangle$ will denote the evaluation of t at y.
Let $F:X \times X -\!\circ Z$ and $G:X \times X -\!\circ Z$ be defined by $F(x, y) = \langle T(x), y - x \rangle + H(y) - H(x) = \cup\{\langle t, y - x \rangle | t \in T(x)\} + H(y) - H(x)$ and $G(y, x) = \langle T(y), x - y \rangle + H(y) - H(x) = \cup\{\langle t, x - y \rangle | t \in T(x)\} + H(y) - H(x)$. If $D:X -\!\circ Z$ and $C(x) = D(x)$ for all $x \in X$, problems (4)–(7) contain the following Stampachia variational inequality problems

(a) $\bar{x} \in X$ is a solution of (4) if and only if for each $y \in X$, there exists $t \in T(\bar{x})$ such that $\langle t, y - \bar{x} \rangle + H(y) - H(\bar{x}) \notin -\mathrm{int}\, \mathrm{C}(\bar{x})$;
(b) $\bar{x} \in X$ is a solution of (5) if and only if $\langle t, y - \bar{x} \rangle + H(y) - H(\bar{x}) \in C(\bar{x})$ for all $t \in T(\bar{x})$ and for all $y \in X$;
(c) $\bar{x} \in X$ is a solution of (6) if and only if for each $y \in X$, there exists $t \in T(\bar{x})$ such that $\langle t, y - \bar{x} \rangle + H(y) - H(\bar{x}) \in \mathrm{C}(\bar{x})$;

(d) $\bar{x} \in X$ is a solution of (7) if and only if for all $y \in X$, and for all $t \in T(\bar{x})$, $\langle t, y-\bar{x}\rangle + H(y) - H(\bar{x}) \notin -\mathrm{int}\,\mathrm{C}(\bar{x})$. Problems (8)–(11) contain the following Minty variational inequality problems:

(e) $\bar{x} \in X$ is a solution of (8) if and only if for each $y \in X$, there exists $t \in T(y)$ such that $\langle t, \bar{x}-y\rangle + H(y) - H(\bar{x}) \notin -\mathrm{int}\,\mathrm{C}(\bar{x})$;

(f) $\bar{x} \in X$ is a solution of (9) if and only if $\langle t, \bar{x}-y\rangle + H(y) - H(\bar{x}) \in \mathrm{C}(\bar{x})$ for all $y \in X$ and for all $t \in T(y)$;

(g) $\bar{x} \in X$ is a solution of (10) if and only if for each $y \in X$, there exists $t \in T(y)$ such that $\langle t, \bar{x}-y\rangle + H(y) - H(\bar{x}) \in -C(\bar{x})$;

(h) $\bar{x} \in X$ is a solution of (11) if and only if $\langle t, \bar{x}-y\rangle + H(y) - H(\bar{x}) \notin \mathrm{int}\,\mathrm{C}(\bar{x})$ for all $y \in X$, $t \in T(y)$.

Konnov [15] studied the case of (a) and (h) when $C(x) = C$ for all $x \in X$, and T is a single-valued function.

(iv) If X is a nonempty subset of a topological space E, $T: X \multimap L(E, \mathbb{R}) = E^*$, $f: X \times X \times X \to \mathbb{R}$ is a function, $h: X \times Y \to \mathbb{R}$ be a function. Let $F: X \times X \to \mathbb{R}$ be defined by $F(x, y) = f(x, y, T(x))$. Then $\bar{x} \in X$ is a solution of (1) if and only if $f(\bar{x}, y, v) \geqslant 0$ for all $v \in T(\bar{x})$. We see that $(\bar{x}, y)$ is a feasible solution of the following Problem:

$$\min_{(x,y)} h(x, y) \text{ such that } x \in X, y \in X, f(x, y, v) \geqslant 0 \text{ for all } v \in T(x).$$

(v) If X is a nonempty subset of E, $Z = \mathbb{R}$, $F: X \times Y \to \mathbb{R}$ be a function, then problems (4)–(7) relate to the existence of feasible solution of the following semi-infinite problem [8]:

$$\min_{x} h(x) \text{ such that } F(x, y) \geqslant 0 \text{ for all } y \in X, \text{ where h}: X \to \mathbb{R} \text{ is a function.}$$

The relevance of the Minty problem to applications was pointed out in [11, 12]. Konnov et al. [13] studied problems (4) and (8) when $G = F$ and $C = D$, Konnov and Yao [14] studied problems (4) and (8) when $G = F$ are single vector valued functions and $C(x) = D(x) = C$.

The existence of solutions to Minty variational inequality problem, with additional continuity imposed on G, implies that the Stampachia variational inequality problem is also solvable. Moreover, this property allows one to construct rather simple iterative solution; e.g., see Ref. [13] and references theorem.

It is well-known that, in contrast to the Stampachia problem, compactness and convexity of X and the continuity assumption does not guarantee the existence of a solution of Minty variational inequality problem. Indeed certain kind of generalized monotonicity assumption is needed to ensure the existence of Minty variational inequality problem [15]. Problems

(8)–(11) are generalizations of Minty variational inequality problem. We need certain generalized monotonicity assumptions to guarantee the existence of solutions to these types of problems. See for example [5,11–13,17, 19]. Therefore, the study of generalized monotonicity has became one of the important subjects in the research of equilibrium problems and Minty type variational inequality problems. See for example [12,13,17].

It is easy to see that under certain pseudomonotone condition that if $\bar{x}$ is a solution of (4), then $\bar{x}$ is a solution of (8), see for example [14,19] and references therein. Konnov and Yao [14] and Lin et al. [19] gives the sufficient conditions that (4) and (8) have the same solutions.

In this paper, we define strong type I, strong type II, weak type I, and weak type II maximal $G_{D,C}$-pseudomonotone. In [19], we studied the sufficient condition of weak type I maximal $G_{D,C}$-pseudomonotone and establish the relationship between the solution sets WIK^p and $\mathrm{WIK}^p_{G,D}$. From this relation, we establish the existence theorem of (GVEP) (4). In this paper, we continue to establish the sufficient conditions of strong type I maximal $G_{D,C}$-pseudomonotone, strong type II maximal $G_{D,C}$-pseudomonotone and weak type II maximal $G_{D,C}$-pseudomonotone and the relationships between the solution sets WIK^p, $\mathrm{SIK}^d_{G,D}$, SIK^p, $\mathrm{SIIK}^p_{G,D}$, WIIK^p and $\mathrm{WIIK}^d_{G,D}$. We first establish the existence of solutions of (GVEP) (8–11). From these relationships of solution sets, we establish the existence theorems of (GVEP) (4–7). As applications of our existence theorems of (GVEP) (4–7), we study the existence theorems of the following two types of generalized vector semi-infinite programming:

Type 1: $\mathrm{wMin}_{x\in A} h(x)$, where

$$A=\{x\in X: F(x,y)\in C \quad \text{for all } y\in X\}; \tag{12}$$

Type 2: $\mathrm{wMin}_{x\in A} h(x)$, where

$$A=\{x\in X: F(x,y)\notin(-\operatorname{int}C) \quad \text{for all } y\in X\}, \tag{13}$$

and X is a compact convex subset of a real t.v.s., Z a real t.v.s. ordered by a proper closed convex cone with $\operatorname{int}C\neq\emptyset$, $h{:}X\to Z$ is a function and $T{:}X - L(E,Z)$, $F{:}X\times X -\!\circ Z$ are multivalued maps.

If $Z=\mathbb{R}$, $C=[0,\infty)$, $h{:}X\to\mathbb{R}$ and F is a real single-valued function, the above two types of generalized vector semi-infinite programming will be reduced to the following semi-infinite programming:

$$\mathrm{Min}_{x\in A} h(x), \quad \text{where } A=\{x\in X{:}F(x,y)\geqslant 0 \text{ for all } y\in X\}.$$

2. Preliminaries

Let X and Y be nonempty sets. A multivalued map $T:X \multimap Y$ is a function from X into the power set of Y. Let $x \in X$ and $y \in Y$, we denote $x \in T^{-}(y)$ if and only if $y \in T(x)$. The graph of T is denoted by $G_r(T)$, where

$$G_r(T) = \{(x, y) \in X \times Y : x \in X, y \in T(x)\}.$$

Let X and Y be topological spaces. A multivalued map $T:X \multimap Y$ is called

(i) closed if $G_r(T)$ is closed subset of $X \times Y$;
(ii) upper semicontinuous (in short u.s.c.) if for every $x \in X$ and every open set V in Y with $T(x) \subset V$, there exists a neighborhood $W(x)$ of x such that $T(u) \subset V$ for all $u \in W(x)$;
(iii) lower semicontinuous (in short l.s.c.) if for every $x \in X$ and every open set V in Y with $T(x) \cap V \neq \emptyset$, there exists a neighborhood $W(x)$ of x such that $T(u) \cap V(y) \neq \emptyset$ for all $u \in W(x)$;
(iv) continuous if T is both u.s.c. and l.s.c;
(v) compact if $\overline{T(X)}$ is a compact set in Y.

A convex space [16] X is a nonempty convex set in a vector space with any topology that induces the Euclidean topology on the convex hulls of its finite subset.

Let X be a convex space and Y be a Hausdorff topological space. If S, T: $X \multimap Y$ are multivalued maps such that $T(\mathrm{co}N) \subseteq S(N)$ for each $N \in \langle X \rangle$, then S is said to be generalized KKM mapping w.r.t. T [7]. The multivalued map $T:X \multimap Y$ is said to have the KKM property [7] if $S:X \multimap Y$ is a generalized KKM w.r.t. T such that the family $\{\overline{S(x)}:x \in X\}$ has the finite intersection property.

We denote by $\mathrm{KKM}(X, Y)$ the family of all multivalued maps having the KKM property [7], we denote by $K(X, Y)=\{T|T:X \multimap Y$ is an u.s.c. multivalued map with nonempty compact convex values$\}$. Then $K(X, Y) \subseteq \mathrm{KKM}(X, Y)$; see for example [7].

Let X be a convex space and Z be a topological vector space (in short t.v.s.). Let F, $G:X \times X \multimap Z$, $C : X \multimap Z$ and $D:X \multimap Z$ such that for each $x \in X$, $C(x)$ and $D(x)$ are proper closed convex cones with int $C(x) \neq \emptyset$ and $\mathrm{int}D(x) \neq \emptyset$. Then F is called

(a) strong type I $G_{D,C}$-pseudomonotone if for all x, $y \in X$, $F(x, y) \subset C(x)$ implies $G(y, x) \subseteq (-D(x))$;
(b) strong type II $G_{D,C}$-pseudomonotone if for all x, $y \in X$, $F(x, y) \cap C(x) \neq \emptyset$ implies $G(y, x) \cap (-D(x)) \neq \emptyset$;
(c) weak type I $G_{D,C}$-pseudomonotone if for all x, $y \in X$, $F(x, y) \nsubseteq (-\mathrm{int}C(x))$ implies $G(y, x) \nsubseteq (\mathrm{int}D(x))$;

(d) weak type II $G_{D,C}$-pseudomonotone if for all $x,\ y \in X,\ F(x, y) \cap (-\text{int}C(x)) = \emptyset$ implies $G(y, x)\cap)(\text{int}D(x)) = \emptyset$;

(e) strong type explicitly $\delta(C_x)$-quasiconvex if for all $y_1,\ y_2 \in X$ and $t \in (0, 1)$, we have either $F(y_t, y_1) \subseteq F(y_t, y_t) + C(y_1)$ or $F(y_t, y_2) \subseteq F(y_t, y_t) + C(y_1)$ and in case $F(y_t, y_1) - F(y_t, y_2) \nsubseteq (-C(y_1))$ for all $t \in (0, 1)$, we have $F(y_t, y_1) \subseteq F(y_t, y_t) + \text{int}C(y_1)$, where $y_t = ty_1 + (1-t)y_2$.

(f) weak type explicitly $\delta(C_x)$-quasiconvex if for all $y_1,\ y_2 \in X$ and $t \in (0, 1)$, we have either $F(y_t, y_1) \subseteq F(y_t, y_t) + C(y_1)$ or $F(y_t, y_2) \subseteq F(y_t, y_t) + C(y_1)$ and in case $[F(y_t, y_1) - F(y_t, y_2)] \cap (\text{int}C(y_1)) \neq \emptyset$. for all $t \in (0, 1)$, we have $F(y_t, y_1) \subseteq F(y_t, y_t) + \text{int}C(y_1)$.

(g) explicitly $\delta(C_x)$-quasiconvex [14], if for all $y_1,\ y_2 \in X$, and $t \in (0, 1)$, either $F(y_t, y_1) \subseteq F(y_t, y_t) + C(y_1)$ or $F(y_t, y_2) \subseteq F(y_t, y_t) + C(y_2)$ and in case $F(y_t, y_1) - F(y_t, y_2) + \text{int}C(y_1)$ for all $t \in (0, 1)$, we have $F(y_t, y_1) \subseteq F(y_t, y_t) + \text{int}C(y_1)$.

(h) strong type I maximal $G_{D,C}$-pseudomonotone, if F is strong type I $G_{D,C}$-pseudomonotone and for all $x,\ y \in K,\ G(z, x) \subseteq (-D(x))$ for all $z \in (x, y]$ implies $F(x, y) \subseteq C(x)$, where $(x, y] = \{z \in K : z = ty + (1-t)x,\ t \in (0, 1]\}$ is a line segment in X joining x and y but not x;

(i) strong type II maximal $G_{D,C}$-pseudomonotone, if F is strong type II $G_{D,C}$-pseudomonotone and for all $x,\ y \in X,\ G(z, x) \cap (-D(x)) \neq \emptyset$ for all $z \in (x, y]$ implies $F(x, y) \cap C(x) \neq \emptyset$;

(j) weak type I maximal $G_{D,C}$-pseudomonotone[19] if F is weak type I $G_{D,C}$-pseudomonotone and for all $x,\ y \in X,\ G(z, x) \nsubseteq \text{int}D(x)$ for all $z \in (x, y]$ implies $F(x, y) \nsubseteq (-\text{int}C(x))$;

(k) weak type II maximal $G_{D,C}$-pseudomonotone if F is weak type II $G_{D,C}$-pseudomonotone and for all $x,\ y \in X,\ G(z, x) \cap (\text{int}D(x)) = \emptyset$ for all $z \in (x, y]$ implies $F(x, y) \cap (-\text{int}C(x)) = \emptyset$.

Let $H: X \multimap Z$, then H is called

(i) C_x-quasiconvex[14] if for all $x,\ y_1,\ y_2 \in X$ and $t \in [0, 1]$, we have either $H(y_1) \subseteq H(ty_1 + (1-t)y_2) + C(x)$ or $H(y_2) \subseteq H(ty_1 + (1-t)y_2) + C(x)$.

(ii) C_x-quasiconcave like if for all $x,\ y_1,\ y_2 \in X$ and $t \in [0, 1]$, we have either $H(ty_1 + (1-t)y_2) \subseteq H(y_1) + C(x)$ or $H(ty_1 + (1-t)y_2) \subseteq H(y_2) + C(x)$;

(iii) Concave, if for all $y_1,\ y_2 \in X$ and $\lambda \in [0, 1]$,

$$\lambda H(y_1) + (1-\lambda)H(y_2) \subseteq H(\lambda y_1 + (1-\lambda)y_2).$$

REMARK. Strong type and weak type explicitly $\delta(C_x)$-quasiconvex are different from explicitly $\delta(C_x)$-quasiconvex as defined in [14].

DEFINITION 2.1. Let Z be a real t.v.s., C a convex cone in Z with $\text{int}C \neq \emptyset$, and A a nonempty subset of Z.

(i) Let z_1, $z_2 \in A$, we denote $z_1 < z_2$ if $z_2 - z_1 \in \mathrm{int}C$.
(ii) A point $\bar{y} \in A$ is called a weakly vector minimal point of A if for any $y \in A$, $y - \bar{y} \notin (-\mathrm{int}C)$. The set of weakly vector minimum point of A is denoted by wMin A.

DEFINITION 2.2. Let X be a convex space and Z a real t.v.s. and C a convex cone in Z with $\mathrm{int}C \neq \emptyset$. Let $f: X \to Z$ be a function. f is said to be C-l.s.c. on X if for any $z \in Z$, $\{x \in X, f(x) > z\}$ is open.

LEMMA 2.1. [21]. *Let T be a multivalued map of a topological space X into a topological space Y. Then T is l.s.c. at $x \in X$ if and only if for any net $\{x_\alpha\}$ in X converging to x, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ for every α and y_α converging to y.*

THEOREM 2.2. [4]. *Let X and Y be Hausdorff topological spaces, T: $X -\circ Y$ be a multivalued map.*

(i) *If $T: X -\circ Y$ is an u.s.c. multivalued map with closed values, then T is closed.*
(ii) *If X is a compact and $T: X -\circ Y$ is an u.s.c. multivalued map with compact values, then $T(X)$ is compact.*

THEOREM 2.3. [19]. *Let Y be a convex space, X a topological space, and $T \in \mathrm{KKM}(Y, X)$. Let $P: X -\circ Y$ be a multivalued map such that for each $x \in X$, $P(x)$ is convex, $X = \cup\{intP^-(y) : y \in Y\}$, and for each compact subset A of X, $\overline{T(A)}$ is compact. Assume that there exist a nonempty compact subset K of X and for each $N \in \langle Y \rangle$, a compact convex subset L_N of Y containing N such that $T(L_N) \backslash K \subseteq \cup\{intP^-(y) : y \in L_N\}$. Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in P(\bar{x})$.*

THEOREM 2.4. [22]. *Let X be a nonempty compact convex subset of a Hausdorff t.v.s. and $F: X -\circ X$ be a multivalued map. Suppose that*

(i) *for all $x \in X, x \notin F(x)$ and $F(x)$ is convex;*
(ii) *for all $y \in X$, $F^{-1}(y)$ is open.*

Then there exists $\bar{x} \in X$ such that $F(\bar{x}) = \emptyset$.

3. Existence theorems of maximal pseudomonotonicity

In this section, we will establish the existence theorems of different kinds of maximal pseudomonotonicity.

THEOREM 3.1. *Let X be a convex subset of a Hausdorff t.v.s., Z a Hausdorff t.v.s. $C, D{:}X -\circ Z$ be multivalued maps such that for each $x \in X$, $C(x)$ and $D(x)$ are proper closed convex cones with int $C(x) \neq \emptyset$. Let $F, G{:}X \times X -\circ Z$ be multivalued maps such that*

(i) *for all $x, y \in X$, $F(y, y) \subseteq C(x)$;*
(ii) *F is strong explicitly $\delta(C_x)$-quasiconvex and strong type I $G_{D,C}$-pseudomonotone;*
(iii) *for all $x, y \in X$, $F(y, x) \nsubseteq -C(x)$ implies $G(y, x) \nsubseteq -D(x)$; and*
(iv) *for each fixed $x_1, x_2, y \in X$, and $t \in [0, 1]$, the multivalued map $t -\circ F(tx_2 + (1-t)x_1, y)$ is l.s.c. at 0+.*

Then F is strong type I maximal $G_{D,C}$-pseudomonotone.

Proof. Suppose that $x, y \in X$ and $G(z, x) \subseteq -D(x)$ for all $z \in (x, y]$. We want to show that $F(x, y) \subseteq C(x)$. Suppose that $F(x, y) \nsubseteq C(x)$. Then $F(x, y) \cap [Z \backslash C(x)] \neq \emptyset$. Since $C(x)$ is closed for each $x \in X$ and (iv), for each fixed $x, y \in X$, there exists $\alpha \in (0, 1)$ such that

$$F(x_\alpha, y) \cap [Z \backslash C(x)] \neq \emptyset \tag{14}$$

where $x_\alpha = \alpha y + (1-\alpha)x$. By assumption,$F$ is strong explicitly $\delta(C_x)$-quasiconvex, for each $\alpha \in [0, 1]$, either $F(x_\alpha, y) \subseteq F(x_\alpha, x_\alpha) + C(x) \subseteq C(x) + C(x) \subseteq C(x)$ or $F(x_\alpha, x) \subseteq F(x_\alpha, x_\alpha) + C(x) \subseteq C(x)$.
The first relation contradicts with (12). Thus

$$F(x_\alpha, x) \subseteq C(x). \tag{15}$$

Hence $F(x_\alpha, x) - F(x_\alpha, y) \nsubseteq -C(x)$ follows from (14) and (15). Therefore,

$$F(x_\alpha, x) \subseteq F(x_\alpha, x_\alpha) + \mathrm{int}C(x) \subseteq C(x) + \mathrm{int}C(x) \subseteq \mathrm{int}C(x). \tag{16}$$

By assumption, $C(x)$ is proper for each $x \in X$, $-C(x) \cap \mathrm{int}C(x) = \emptyset$. By (16), $F(x_\alpha, x) \nsubseteq -C(x)$. By (iii), $G(x_\alpha, x) \nsubseteq -D(x)$. This contradicts with $G(z, x) \subseteq -D(x)$ for all $z \in (x, y]$. By assumption, F is also strong type I $G_{D,C}$-pseudomonotone. Therefore, F is strong type I maximal $G_{D,C}$- pseudomonotone. □

PROPOSITION 3.2. *Under the assumption of Theorem 3.1, then* $\mathrm{SIK}^p = \mathrm{SIK}^d_{G,D}$.

Proof. By assumption, F is strong type I $G_{D,C}$-pseudomonotone, it follows that $\mathrm{SIK}^p \subseteq \mathrm{SIK}^d_{G,D}$. We want to show that $\mathrm{SIK}^d_{G,D} \subseteq \mathrm{SIK}^p$.

Let $\bar{x} \in \mathrm{SIK}^d_{G,D}$, then $G(y, \bar{x}) \subseteq -D(x)$ for all $y \in X$. For any fixed $y \in X$, $(\bar{x}, y] \subseteq X$. Therefore $G(z, \bar{x}) \subseteq -D(x)$ for all $z \in (\bar{x}, y]$. By Theorem 3.1,

F is strong type I maximal $G_{D,C}$-pseudomonotone, $F(\bar{x}, y) \subseteq C(\bar{x})$. Hence $\bar{x} \in$ SIK^p and SIK$^d_{G,C} \subseteq$ SIK^p.We show that SIK$^p_{G,D}$ = SIK^p. □

For the particular case of Theorem 3.1, we have the following Corollary.

COROLLARY 3.3. *Let X, Z, C be the same as Theorem 3.1. Let $F{:}X \times X \multimap Z$ be a multivalued map with nonempty values such that*

(i) *for all $x, y \in X$, $F(y, y) \subseteq C(x)$ and $F(y, x) = -F(x, y)$;*
(ii) *F is strong explicitly $\delta(C_x)$-quasiconvex;*
(iii) *for all $y \in X$, the multivalued map $x \multimap F(x, y)$ is l.s.c.*

Then F is strong maximal $G_{D,C}$-pseudomonotone and SIK^p= SIK^d_C.

Proof. Let $G{:}X \times X \multimap Z$ and $C{:}X \multimap Z$ be defined by $G(x, y) = F(x, y)$ and $D(x) = C(x)$. Then Corollary 3.3 follows from Theorem 3.1. □

With the same argument as in Theorem 3.1 and Proposition 3.2, we have the following theorems.

THEOREM 3.4. *Let X, Z, C, D be the same as Theorem 3.1. Let $F, G{:}X \times X \multimap Z$ be multivalued maps such that*

(i) *for all $x, y \in X$, $F(y, y) \subseteq C(x)$;*
(ii) *F is strong explicitly $\delta(C_x)$-quasiconvex and strong type II $G_{D,C}$-pseudomonotone;*
(iii) *for each $x, y \in X$, $F(y, x) \cap (-C(x)) = \emptyset$ implies*

$$G(y, x) \cap (-D(x)) = \emptyset; \text{ and}$$

(iv) *$x \multimap F(x, y)$ is u-hemicontinuous for each $y \in X$;*

Then F is strong type II maximal $G_{D,C}$-pseudomonotone and SIIK^p = SIIK$^d_{G,D}$.

THEOREM 3.5. *Let X, Z, C and D be the same as Theorem 3.1. Let $F, G{:}X \times X \multimap Z$ be multivalued maps with nonempty values such that*

(i) *for all $x, y \in X$, $F(y, y) \subseteq C(x)$;*
(ii) *F is weak explicitly $\delta(C_x)$-quasiconvex and weak type II $G_{D,C}$-pseudomonotone;*
(iii) *for each $x, y \in X$, $F(y, x) \cap (intC(x)) \neq \emptyset$ implies*

$$G(y, x) \cap (\mathrm{int}D(x)) \neq \emptyset; \text{ and}$$

(iv) *for all* $x_1, x_2, y \in X$ *and* $t \in [0, 1]$,

$$t \multimap F(tx_2 + (1-t)x_1, y) \text{ is l.s.c. at } 0.$$

Then F *is weak type II maximal* $G_{D,C}$*-pseudomonotone and* $\mathrm{WIIK}^p = \mathrm{WIIK}^d_{G,D}$.

4. Existence theorems of generalized vector equilibrium problems

As applications of results in Section 3, we establish some existence theorems of generalized vector equilibrium problems.

THEOREM 4.1. *Let* X *be a closed convex subset of a Hausdorff t.v.s.,* Z *a Hausdorff t.v.s. Let* $T \in$ KKM (X, X) *such that for each compact subset* A *of* X, $\overline{T(A)}$ *is compact,* $C, D{:}X \multimap Z$ *be multivalued maps such that for each* $x \in X$, $C(x)$ *and* $D(x)$ *are proper closed convex cones with* $intC(x) \neq \emptyset$. *Let* $F, G : X \times X \multimap Z$ *be multivalued maps such that*

(i) *for all* $y \in X, x \in T(y)$, $F(x, y) \subseteq C(x)$;
(ii) *for each* $x \in X$, $y \multimap G(x, y)$ *is l.s.c. and* $D{:}X \multimap Z$ *is u.s.c.;*
(iii) *for each* $x \in X$, $y \multimap G(y, x)$ *is* D_x*-quasiconcave;*
(iv) F *is strong type I maximal* $G_{D,C}$*-pseudomonotone;*
(v) *there exists a nonempty compact subset* K *of* X *such that for each* $N \in \langle X \rangle$, *there exists a compact convex subset* L_N *of* X *containing* N *such that for each* $x \in T(L_N)\backslash K$, *there exists a* $y \in L_N$ *such that* $G(y, x) \nsubseteq -D(x)$.

Then there exists $\bar{x} \in X$ *such that* $F(\bar{x}, y) \subseteq C(\bar{x})$ *for all* $y \in X$ *and* SIK^p=$\mathrm{SIK}^d_{G,D}$.

Proof. SIK^p=$\mathrm{SIK}^d_{G,D}$ follows from Proposition 3.2. It suffices to show that $\mathrm{SIK}^d_{G,D} \neq \emptyset$.Let $P : X \multimap X$ be defined by

$$P(x) = \{y \in X{:}G(y, x) \nsubseteq -D(x)\}.$$

Suppose to the contrary that $\mathrm{SIK}^d_{G,D} = \emptyset$.Then for each $x \in X$, there exists a $y \in X$ such that $G(y, x) \nsubseteq -D(x)$. Therefore, $P(x) \neq \emptyset$ for each $x \in X$. $[P^-(y)]^c = \{x \in X{:}G(y, x) \subseteq -D(x)\}$ is closed. Indeed, let $u \in \overline{[P^-(y)]^c}$, then there exists a net $\{x_\alpha\}$ in $[P^-(y)]^c$ such that $x_\alpha \to u$. Therefore, $x_\alpha \in X$ and $G(y, x_\alpha) \subseteq -D(x_\alpha)$. Let $z \in G(y, u)$. Since for each fixed $y \in X, x \multimap G(y, x)$ is l.s.c. and $x_\alpha \to u$, it follows from Lemma 2.1 that there exists a net $\{z_\alpha\}$ in $G(y, x_\alpha)$ such $z_\alpha \to z$. We see $-z_\alpha \in -G(y, x_\alpha) \subseteq D(x_\alpha)$. By assumption,

D is u.s.c. with closed values, it follows from Theorem 2.2 that D is closed. Therefore, $-z \in D(u)$ and $G(y,u) \subseteq -D(u)$. Since $x_\alpha \in X$ and $x_\alpha \to u$, $u \in X$. This shows that $u \in [P^-(y)]^c$ and $[P^-(y)]^c$ is a closed set for each $y \in X$. Hence $P^-(y)$ is open for each $y \in X$. For each $x \in X$, $P(x)$ is convex. Indeed, let $y_1, y_2 \in P(x)$ and $\lambda \in [0,1]$, then $y_1, y_2 \in X$ and $G(y_i, x) \nsubseteq -D(x)$ for $i = 1, 2$. Since X is convex, $\lambda y_1 + (1-\lambda) y_2 \in X$. We want to show that $G(\lambda y_1 + (1-\lambda)y_2, x) \nsubseteq -D(x)$ for all $\lambda \in [0,1]$. Suppose to the contrary that there exists $\lambda_0 \in (0,1)$ such that $G(\lambda_0 y_1 + (1-\lambda_0)y_2, x) \subseteq -D(x)$. By assumption, $y -\circ G(y,x)$ is C_x-quasiconcave for each $x \in X$. Either $G(y_1,x) \subseteq G(\lambda_0 y_1 + (1-\lambda_0)y_2, x) - D(x) \subseteq -D(x) - D(x) \subseteq -D(x)$ or $G(y_2,x) \subseteq G(\lambda_0 y_1 + (1-\lambda_0)y_2, x) - D(x) \subseteq -D(x) - D(x) \subseteq -D(x)$. This is a contradiction. This shows that $G(\lambda y_1 + (1-\lambda)y_2, x) \nsubseteq -D(x)$ for all $\lambda \in [0,1]$ and $\lambda y_1 + (1-\lambda) y_2 \in P(x)$. Therefore, $P(x)$ is convex for each $x \in X$. Since $P(x) \neq \emptyset$ for all $x \in X$ and $P^-(y)$ is open for all $y \in X$, $X = \cup\{P^-(y): y \in X\} = \cup\{int P^-(y): y \in X\}$. By (v), $T(L_N)\backslash K \subseteq \cup\{P^-(y): y \in L_N\} = \cup\{int P^-(y): y \in L_N\}$. By Theorem 2.3 that there exists $\bar{x} \in X, \bar{y} \in X$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in P(\bar{x})$. By (i), $F(\bar{x}, \bar{y}) \subseteq C(\bar{x})$ and $G(\bar{y}, \bar{x}) \nsubseteq -D(\bar{x})$. By assumption, F is strong type I maximal $G_{D,C}$-pseudomonotone. Therefore, $F(\bar{x}, \bar{y}) \subseteq C(\bar{x})$ implies $G(\bar{y}, \bar{x}) \subseteq -D(\bar{x})$. This is a contradiction. Therefore, $\mathrm{SIK}^d_{G,D} \neq \emptyset$ and $\mathrm{SIK}^p \neq \emptyset$. Hence there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \subseteq C(\bar{x})$ for all $y \in X$. $\square$

For the special case of Theorem 4.1, we have the following Corollary.

COROLLARY 4.2. *Let X be a closed convex subset of a Hausdorff t.v.s., Z a Hausdorff t.v.s. Let $C: X -\circ Z$ be a multivalued map such that for each $x \in X$, $C(x)$ is a proper closed convex cone with $int C(x) \neq \emptyset$. Let $F: X \times X -\circ Z$ be a multivalued map such that*

(i) *$F(x,x) \subseteq C(x)$ and $F(y,x) = -F(x,y)$ for all $x, y \in X$;*
(ii) *for each $x \in X$, $y -\circ F(x,y)$ is l.s.c. and $C: X -\circ Z$ is u.s.c.;*
(iii) *for each $x \in X$, $y -\circ F(y,x)$ is C_x-quasiconcave;*
(iv) *there exists a nonempty compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that for each $x \in L_N \backslash K$, there exists a $y \in L_N$ such that $F(y,x) \nsubseteq -C(x)$.*

Then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \subseteq C(\bar{x})$ for all $y \in X$.

Proof. Let $G: X \times X -\circ Z$ and $D: X -\circ Z$ be defined by $G(x,y) = F(x,y)$ and $D(x) = C(x)$ for $x, y \in X$. By (i), it is easy to see that F is strong type I $F_{C,C}$-pseudomonotone. If for each $x, y \in X$, $F(z, x) \subseteq -C(x)$ for all $z \in (x, y]$. By (i), $F(x, y) = -F(y, x) \subseteq C(x)$. Therefore, F is strong type I $F_{C,C}$-pseudomonotone, F is strong type I maximal $F_{C,C}$-pseudomonotone. Let $T{:}X -\!\circ X$ be defined by $T(x) = \{x\}$ for $x \in X$. Then $T \in$ KKM (X, X) and for each compact subset M of X. $T(M)$ is a compact subset of X. Then the conclusion of Corollary 4.2 follows from Theorem 4.1. □

COROLLARY 4.3. *In Theorem 4.1, if the conditions $T \in$ KKM (X, X) and for each compact subset A of $X, \overline{T(A)}$ is compact are replaced by $T{:}X -\!\circ X$ is an u.s.c. multivalued map with nonempty compact convex values. Then the conclusion of Theorem 4.1 still holds.*

Proof. By assumption, $T \in K(X \times X) \subseteq \mathrm{KKM}(X, X)$; see for example [7]. Let A is a compact subset of X, then by Theorem 2.2 that $T(A)$ is compact and $\overline{T(A)}$ is compact. The conclusion of Corollary 4.3 follows from Theorem 4.1. □

Applying Theorem 3.4 and following the same argument as in Theorem 4.1, we have the following theorem.

THEOREM 4.4. *Let X, Z, C, T and D be the same as Theorem 4.1. Let $F, G{:}X \times X -\!\circ Z$ be multivalued maps with nonempty values such that*

(i) *for all $y \in X, x \in T(y)$, $F(x, y) \cap C(x) \neq \emptyset$;*
(ii) *D is a closed multivalued map and for each $y \in X, x -\!\circ G(y, x)$ is a closed and compact multivalued map;*
(iii) *for each $x \in X, y -\!\circ G(y, x)$ is D_x-quasiconcave-like; that is, if $y_1, y_2 \in X, \lambda \in [0, 1]$, either $G(\lambda y_1 + (1-\lambda)y_2, x) \subseteq G(y_1, x) + D(x)$* or *$G(\lambda y_1 + (1-\lambda)y_2, x) \subseteq G(y_2, x) + D(x)$;*
(iv) *F is strong type II maximal $G_{D,C}$-pseudomonotone;*
(v) *there exists a nonempty compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that for each $x \in T(L_N)\backslash K$, there exists a $y \in L_N$ with $G(y, x) \cap (-D(x)) = \emptyset$.*

Then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \cap C(\bar{x}) \neq \emptyset$ for all $y \in X$ and $\mathrm{SIIK}^p = \mathrm{SIIK}^d_{G,D} \neq \emptyset$.

Applying Theorems 2.3 and 3.5 and following the same arguments as in Theorem 4.1, we have the following theorem.

THEOREM 4.5. *Let X, Z, T, C and D be the same as Theorem 4.1. Let $F, G : X \times X -\!\circ Z$ be multivalued maps such that*

(i) *for all $y \in X, x \in T(y)$, $F(x, y) \subseteq C(x)$;*

(ii) *for each* $x \in X$, $y -\circ G(x, y)$ *is l.s.c. and* $W : X -\circ Z$ *is u.s.c., where* $W(x) = Z \backslash (int D(x))$;
(iii) $y -\circ G(y, x)$ *is concave;*
(iv) F *is weak type II maximal* $G_{D,C}$*-pseudomonotone; and*
(v) *there exists a nonempty compact subset* K *of* X *such that for each* $N \in \langle X \rangle$, *there exists a compact convex subset* L_N *of* X *containing* N *such that for each* $x \in T(L_N) \backslash K$, *there exists* $y \in L_N$ *such that* $G(y, x) \cap (int D(x)) \neq \emptyset$.

Then there exists $\bar{x} \in X$ *such that* $F(\bar{x}, y) \cap (-int C(\bar{x})) = \emptyset$ *for all* $y \in X$ *and* $\mathrm{WIIK}^p = \mathrm{WIIK}^d_{G,D} \neq \emptyset$.

For the GVEP(4), we have the following Theorem which is different from Theorem 3.1 [19].

Applying Theorem 2.3 and following the same arguments as in Theorem 4.1, we have the following theorem.

THEOREM 4.6. *Let* X, Z, T, C *and* D *be the same as Theorem 4.1. Let* $F, G{:}X \times X -\circ Z$ *be multivalued maps such that*

(i) *for all* $y \in X, x \in T(y)$, $F(x, y) \nsubseteq (-int C(x))$;
(ii) $W{:}X -\circ Z$ *is an u.s.c. multivalued map and* $x -\circ G(y, x)$ *is a closed and compact multivalued map for each fixed* $y \in X$, *where* $W(x) = Z \backslash (int D(x))$;
(iii) *for each fixed* $x \in X$, $y -\circ G(y, x)$ *is* D_x*-quasiconcave-like;*
(iv) F *is weak type I maximal* $G_{D,C}$*-pseudomonotone;*
(v) *there exists a nonempty compact subset* K *of* X *such that for each* $N \in \langle X \rangle$, *there exists a compact convex subset* L_N *of* X *containing* N *such that for each* $x \in T(L_N) \backslash K$, *there exists a* $y \in L_N$ *with* $G(y, x) \subset int D(x)$.

Then $\mathrm{WIK}^p = \mathrm{WIK}^d_{G,D} \neq \emptyset$ *and there exists* $\bar{x} \in X$ *such that* $F(\bar{x}, y) \nsubseteq (-int C(\bar{x}))$ *for all* $y \in X$.

5. Applications to semi-infinite programming

As consequences of generalized vector equilibrium problems, we establish the existence theorems of generalized vector semi-infinite programming.

THEOREM 5.1. *Let* X *be a compact convex subset of a Hausdorff t.v.s.,* Z *a real Hausdorff t.v.s. ordered by a proper closed convex cone* C *with* $int C \neq \emptyset$. *Let* $h{:}X \to Z$ *and* $F : X \times X \to Z$ *be functions. Suppose that*

(i) *for all* $x \in X$, $F(x, x) \in C$ *and* $F(y, x) = -F(x, y)$ *for all* $x, y \in X$;

(ii) *for each* $x \in X, y \to F(x, y)$ *is continuous and* $y \to F(y, x)$ *is* C*-quasiconcave;*
(iii) h *is* C*-l.s.c. and* C*-quasiconvex.*

Then there exists $\bar{x} \in X$ *such that* $\bar{x}$ *solve the problem:*

$$\text{wMin}_{x \in A} h(x) \quad \text{where } A = \{x \in X : F(x, y) \in C \text{ for all } y \in X\}.$$

Proof. By (i) and (ii), for each fixed $y \in X, x \to F(x, y) = -F(y, x)$ is continuous. Then it follows from Corollary 4.2 that there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \in C$ for all $y \in X$. This shows that $A \neq \emptyset$. Also, A is closed. Indeed, let $x \in \bar{A}$, then there exists a net $x_\alpha \in A$ such that $x_\alpha \to x$. Then $x_\alpha \in X$ and $F(x_\alpha, y) \in C$ for all $y \in X$. By (ii), $F(x, y) \in C$. We see $x \in X$. This shows A is closed. Hence A is compact. And A is convex. Indeed, if $x_1, x_2 \in A$ and $\lambda \in [0, 1]$, then $x_1, x_2 \in X, F(x_1, y) \in C$ and $F(x_2, y) \in C$ for all $y \in X$.

By (ii), for each $y \in C$ either $F(\lambda x_1 + (1-\lambda)x_2, y) \in F(x_1, y) + C \subseteq C$ or $F(\lambda x_1 + (1-\lambda)x_2, y) \in F(x_2, y) + C \subseteq C$. Therefore, $\lambda x_1 + (1-\lambda)x_2 \in A$, that is A is convex.

Let $H(x, y) = h(y) - h(x)$ and $Q(x) = \{y \in A : H(x, y) \in -intC\}$. By (iii), for each $y \in X, Q^-(y) = \{x \in X : H(x, y) \in -intC\}$ is open. For each $x \in X, Q(x)$ is convex. Indeed, if $y_1, y_2 \in Q(x), \lambda \in [0, 1]$, by (v) either $h(\lambda y_1 + (1-\lambda)y_2) \in h(y_1) - C$ or $h(\lambda y_1 + (1-\lambda)y_2) \in h(y_2) - C$. Therefore, either $H(x, \lambda y_1 + (1-\lambda)y_2) \in H(x, y_1) - C \subseteq -intC$ or $H(x, \lambda y_1 + (1-\lambda)y_2) \in H(x, y_2) - C \subseteq -intC$. This shows that $\lambda y_1 + (1-\lambda)y_2 \in Q(x)$, and $Q(x)$ is convex for each $x \in A$. For each $x \in X, x \notin Q(x)$. Then it follows from Theorem 2.4 that there exists $\bar{x} \in A$ such that $Q(\bar{x}) = \emptyset$. Therefore, $\bar{x}$ is the solution of generalized vector semi-infinite programming:

$$\text{wMin}_{x \in A} h(x) \quad \text{where } A = \{x \in X : F(x, y) \in C \text{ for all } y \in X\}. \qquad \square$$

As a special case of Theorem 5.1, we have the following corollary.

COROLLARY 5.2. *Let* X *be a compact convex subset of a Hausdorff t.v.s.,* $h : X \to \mathbb{R}$ *and* $f, g : X \times X \to \mathbb{R}$ *be functions. Suppose that*

(i) $f(x, x) \geqslant 0$ *for all* $x \in X$ *and* $f(y, x) = -f(x, y)$ *for all* $x, y \in X$*;*
(ii) *for each* $x \in X, y \to f(x, y)$ *is continuous and* $y \to f(y, x)$ *is quasiconcave;*
(iii) h *is l.s.c. and quasiconvex.*

Then there exists $\bar{x} \in X$ *such that* $\bar{x}$ *solves the problem*

$$\text{Min}_{x \in A} h(x), \quad \text{where } A = \{x \in X : f(x, y) \geqslant 0 \quad \text{for all } y \in X\}.$$

Proof. Let $Z=\mathbb{R}$. Then Corollary 5.1 follows from Theorem 5.1. □

THEOREM 5.3. *Let X, Z and C be the same as Theorem 5.1. Let $h:X\to Z$ be a function and $F:X\times X\to Z$ be a function. Suppose that*

(i) *for all $x\in X$, $F(x,x)\notin(-intC)$ and $F(y,x)=-F(x,y)$ for all $x,y\in X$;*
(ii) *for each $x\in X$, $y\to F(y,x)$ is C-quasiconcave and continuous; and*
(iii) *h is C-l.s.c. and C-quasiconcave;*

Then there exists $\bar{x}\in X$ such that $\bar{x}$ is the solution of the following generalized vector semi-infinite program:

$$\mathrm{wMin}_{x\in A}h(x) \quad \text{where } A=\{x\in X:F(x,y)\notin -intC \quad \text{for all } y\in X\}.$$

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